ALL SAINTS' COLLEGE

Mathematics
Specialist

Test 5 2016

Applications of Differentiation

NAME: SOLUTIONS

TEACHER: MLA

Resource Free Section

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Question 1 [~~3~~ marks]

Using the identity $\sin^2\theta + \cos^2\theta = 1$, show that if $x = A\sin(\omega t + \alpha)$, then $v^2 = \omega^2(A^2 - x^2)$

$$x = A\sin(\omega t + \alpha)$$

$$v = A\omega\cos(\omega t + \alpha)$$

$$v^2 = A^2\omega^2\cos^2(\omega t + \alpha) \quad ✓$$

$$= A^2\omega^2[1 - \sin^2(\omega t + \alpha)] \quad ✓$$

$$= A^2\omega^2 - A^2\omega^2\sin^2(\omega t + \alpha)$$

$$= \omega^2[A^2 - A^2\sin^2(\omega t + \alpha)] \quad ✓$$

$$= \omega^2\{A^2 - [A\sin(\omega t + \alpha)]^2\}$$

$$= \omega^2(A^2 - x^2) \quad ■ \quad ✓$$

or $\longrightarrow$ $v^2 = a^2\omega^2\left(1 - \frac{x^2}{A^2}\right)$

$\because \sin(\omega t + \alpha) = \frac{x}{A}$

$= \omega^2(A^2 - x^2)$ ■

or

$$\begin{cases} \cos(\omega t + \alpha) = \dfrac{v}{\omega A} & (1) \\ \sin(\omega t + \alpha) = \dfrac{x}{A} & (2) \end{cases}$$

sub (1) and (2) into

$\sin^2\theta + \cos^2\theta = 1$

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Question 2 [~~4~~ marks]

Solve the $\frac{dy}{dx} - 2y = 12$ if $y = 4$ when $x = 0$ and $y \geq 0$.

$$\frac{dy}{dx} = 2y + 12$$

$$= 2(y + 6)$$

$$\int \frac{1}{y+6}\,dy = \int 2\,dx \quad ✓$$

$$\ln(y+6) = 2x + c \quad ✓$$

$$y + 6 = e^c \cdot e^{2x} \quad ✓$$

sub $(0, 4)$ to determine e^c $\Rightarrow$ $10 = e^c$ ✓

$$y + 6 = 10e^{2x}$$

$$\therefore y = 10e^{2x} - 6 \quad ✓$$

or $\frac{1}{2}\int \frac{2}{2y+12}\,dy = \int 1\,dx$

$$\tfrac{1}{2}\ln(2y+12) = x + c$$

$$\ln(2y+12) = 2x + 2c$$

$$2y + 12 = e^{2c} \cdot e^{2x}$$

sub $(0,4) \Rightarrow 20 = e^{2c}$

$$2y + 12 = 20e^{2x}$$

$$2y = 20e^{2x} - 12$$

$$\therefore y = 10e^{2x} - 6$$

Alternatively:

$\left(\frac{2a}{m}\right)^2 = 4ax$

$\frac{4a^2}{m^2} = 4ax$

$\Rightarrow x = \frac{a}{m^2}$ (2)

now sub (1) and (2) into

$y = mx + c$

to show $c = \frac{a}{m}$.

$\therefore y = mx + \frac{a}{m}$.

Question 3 **[~~4~~ 5 marks]**

Show that the equation of the tangent (with gradient m) to the curve with equation $y^2 = 4ax$ is $y = mx + \frac{a}{m}$

$y^2 = 4ax \Rightarrow 2y\frac{dy}{dx} = 4a$ ✓ (implicitly)

Note. $\frac{dy}{dx} = \frac{2a}{y} = m \Rightarrow y = \frac{2a}{m}$ (1) ✓

Equation of tangent line: $y = mx + c$

$y = \frac{2ax}{y} + c$

$y = \frac{4ax}{2y} + c$ ✓

$y = \frac{y^2}{2y} + c$ ✓ $\because y^2 = 4ax$

$2y = y + 2c$

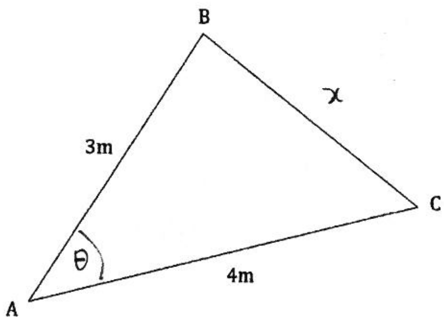
$\frac{y}{2} = c$ ✓

$y = mx + \frac{y}{2} \Rightarrow y = mx + \frac{2a}{2m}$ $\because y = \frac{2a}{m}$ ✓

$\therefore y = mx + \frac{a}{m}$ ∎

Question 4 **[~~5~~ 6 marks]**

In ΔABC, $AB = 3m$, $AC = 4m$ and $\angle BAC = \theta$. If θ is increasing at a rate of π radians per minute, find the rate (in metres per minute), at which the length of the side BC is changing at the instant $\theta = \frac{\pi}{2}$.



Given $\frac{d\theta}{dt} = \pi$ rads/min

Let $x = BC$

RTF: $\frac{dx}{dt}$ when $\theta = \frac{\pi}{2}$

By cosine rule, $x^2 = 9 + 16 - 2(3)(4)\cos\theta$ ✓

Differentiate both sides wrt t:

$2x\frac{dx}{dt} = 24\sin(\theta).\frac{d\theta}{dt}$ //

$\frac{dx}{dt} = \frac{24\sin(\frac{\pi}{2}).\pi}{2(5)}$

$= \frac{24\pi}{10}$

$= \frac{12\pi}{5}$ ✓

when $\theta = \frac{\pi}{2}$

$x^2 = 9 + 16 - 2(3)(4)\cos(\frac{\pi}{2})$

$x^2 = 25$

$x = 5$ ✓ reject $x = -5$ $\because x > 0$

That is, BC is <u>increasing</u> in length at a rate of $\frac{12\pi}{5}$ m/min when $\theta = \frac{\pi}{2}$ ✓

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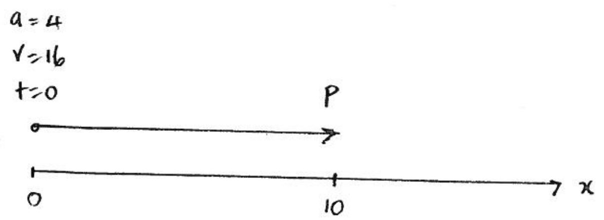
Resource Rich Section

Question 5 **[5 marks]**

Particle P travels along a straight line. At time $t = 0$ seconds, it passes a fixed point O with velocity of 16 ms^{-1} and undergoes constant acceleration of 4 ms^{-2}.

Use calculus to find the velocity of P when it is 10 metres from O.

RTF : $\left.\frac{dx}{dt}\right|_{x=10}$.



$$\frac{d^2x}{dt^2} = 4$$

$$\frac{dx}{dt} = 4t + c$$

when $t = 0$, $\frac{dx}{dt} = 16 \Rightarrow c = 16$

$\therefore \frac{dx}{dt} = 4t + 16$ ✓✓

$$x = 2t^2 + 16t + c$$

when $t = 0$, $x = 0 \Rightarrow c = 0$

$\therefore x = 2t^2 + 16t$ ✓

When $x = 10$, Solve $10 = 2t^2 + 16t$

ClassPad : $t = \sqrt{21} - 4$ or $0.58257...$ ✓

✓

So, when $t = \sqrt{21} - 4$, $\frac{dx}{dt} = 4\sqrt{21}$ or 18.33 m/sec (2dp)

Note I. Use of $v^2 = u^2 + 2as$
$= (16)^2 + 2(4)(10)$
$= 18.3$ m/sec

1 mark only ... RTQ !!

Note II. Use of $s = ut + \frac{1}{2}at^2$ and $v = u + at$

5

Question 6 **[2 & 4 = 6 marks]**

An object travels in a straight line such that its velocity at time t is given by $v = e^{\sin(t)}$ mm/s

(a) Describe the direction of travel for $0 \leq t \leq 10$

$v(0) = 1$ mm/sec $\therefore$ Initially travelling to the right ✓

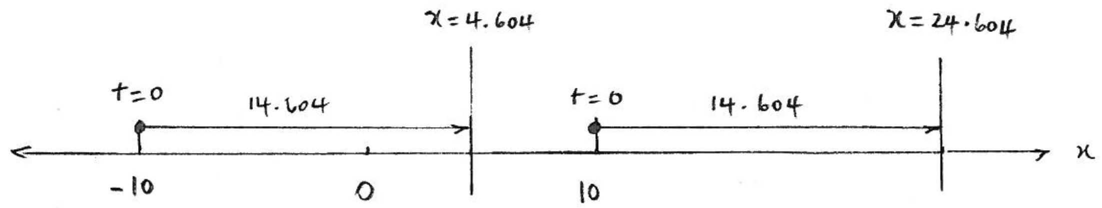
Solve $0 = e^{\sin(t)}$ ... no solution $\Rightarrow$ object does not stop and change direction at any stage. ✓

Conclusion: object travels to right for $0 \leq t \leq 10$.

Note. could also show that distance travelled = displacement for $0 \leq t \leq 10$

i.e. $\int_0^{10} |e^{\sin(t)}| \, dt = \int_0^{10} e^{\sin(t)} \, dt \Rightarrow 14.6039... = 14.6039...$

(b) If the object was 10 mm from a fixed observation point O at t = 0, use calculus to establish its displacement at t = 10 seconds.



Δ Displacement $= \int_0^{10} e^{\sin(t)} \, dt$

$= 14.604$ (3dp) ✓✓

Two possible solutions:

(i) If $x(0) = -10$, $x = 4.604$ mm (3dp) ✓

(ii) If $x(0) = 10$, $x = 14.604$ mm (3dp) ✓

Question 7 **[5, 1, 1 & 2 = 9 marks]**

According to Newton's Law of Cooling, the temperature T°C (Celcius) of a hot brass plate left to cool down satisfies the equation $\frac{dT}{dt} = -k(T - 20)$, where k is a positive constant and t is measured in minutes.

(a) After 20 minutes the temperature of the brass plate is 60°C and after 30 minutes it is 30°C. Express T as a function of t.

$$\int \frac{1}{T-20}\, dT = -k \int 1\, dt \quad ✓$$

$$\ln(T-20) = -kt + c \quad ✓$$

$$T - 20 = e^c \cdot e^{-kt}$$

$$T = 20 + e^c \cdot e^{-kt}$$

sub (20, 60) ⇒ $60 = 20 + e^c \cdot e^{-20k}$ ⇒ $40 = e^c \cdot e^{-20k}$

sub (30, 30) → $30 = 20 + e^c \cdot e^{-30k}$ ⇒ $10 = e^c \cdot e^{-30k}$ //

Solving simultaneously (classpad) : $T = 20 + 640e^{-\frac{\ln(2)}{5}t}$ ✓

$e^c = A = 640$ $\quad$ $k = 0.138629...$

or divide these two equations and solve : $4 = e^{10k}$

$\Rightarrow k = \frac{\ln(2)}{5}$ $\quad \therefore e^c = 640.$

(b) What is the initial temperature of the brass plate?

$T(0) = 660°C$ ✓

(c) What is the final temperature of the brass plate?

As $t \to \infty$, $T \to 20°C$ ✓ (ambient temperature)

(d) How long will it take for the temperature of the brass plate to drop to within 5°C of its final temperature?

Solve $25 = 20 + 640e^{-\frac{\ln(2)}{5}t}$ ✓

ClassPad : $t = 35$

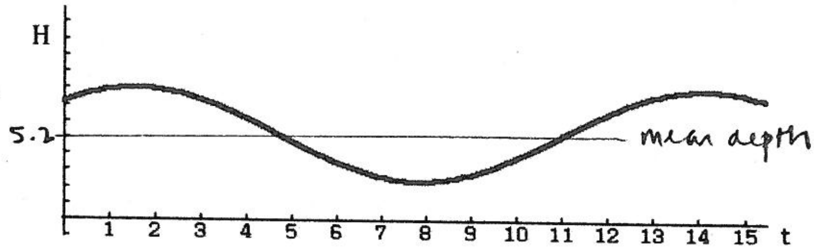
That is, it will take 35 minutes to reach 25°C.

✓

1 3

Question 8 [~~2~~, 1, 1, 1, ~~1~~1 & 3= 10 marks]

At a point in a small bay near Cable Beach in Broome, the depth H (metres) of the water at time t (hours) is given by:



$H = 5.2 + 2\sqrt{2}\sin\left(\frac{\pi}{4} + \frac{t}{2}\right)$

$t = 0$ corresponds to 9.00am.

The sinusoidal motion of the tide shown above can be replicated on your ClassPad.

If the depth of water in the bay is influenced only by the tide,

(a) State the water depth when $t = 0$ 7.2m ✓

(b) What is the water level about which the tide oscillates? 5.2m ✓

(c) What is the exact range of depth in the bay? Amplitude of Motion $= 2\sqrt{2}$

$\therefore$ Range of Depth $= 4\sqrt{2}$ or 5.66m (2dp)

(d) How much time lapses between successive high tides?

Period of Motion $= \frac{2\pi}{\omega}$, where $\omega = \frac{1}{2}$ $\therefore$ 4π or 12.6 hours lapse (1dp) ✓

(e) How fast is the water depth changing at 1:00pm?

$\frac{dH}{dt} = \sqrt{2}\cos\left(\frac{\pi}{4} + \frac{t}{2}\right)$ ✓ $\Rightarrow$ when $t = 4$, $\frac{dH}{dt} = -1.3254\ldots$ ✓

$\therefore$ Depth is decreasing at 1.33m/hour at 1pm (2dp) ✓

(f) Show that the change in water depth is an example of SHM.

change in water depth $= H - 5.2$

$= 2\sqrt{2}\sin\left(\frac{\pi}{4} + \frac{t}{2}\right)$

Let $x \equiv H - 5.2$

$\Rightarrow \quad x = 2\sqrt{2}\sin\left(\frac{\pi}{4} + \frac{t}{2}\right)$

$\dot{x} = \sqrt{2}\cos\left(\frac{\pi}{4} + \frac{t}{2}\right)$

$\ddot{x} = -\frac{\sqrt{2}}{2}\sin\left(\frac{\pi}{4} + \frac{t}{2}\right)$ ✓

$= -\frac{1}{4} \cdot 2\sqrt{2}\sin\left(\frac{\pi}{4} + \frac{t}{2}\right)$ ✓

$= -\frac{1}{4}x$ ✓

$\therefore \ddot{x} = -\omega^2 x$, where $\omega = \frac{1}{2}$ ■